NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

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Reflection and Refraction of Electromagnetic Waves by a Dielectric Slab between Dielectric Media

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Abstract

The general problem considered is the reflection and refraction of an arbitrary electromagnetic wave by a dielectric slab between two different media. Explicit expressions are obtained for the radiation field in terms of the incident radiation field. These expressions are valid when the point of observation is not too near the plane interface. The method is extended to cover the case of one surface deviating slightly from a plane.

Table of Contents

		Page
1.	Introduction	1
2.	Mathematical formulation	2
3.	The determination of the Green's functions	7
4.	Solution of the integral equations	8
5.	The solution of special problems	9
6.	The asymptotic forms of the field	11
7.	Solution for one surface deviating slightly from a plane	14
References		16

1. Introduction

The object of the present note is to present a unified method of treating the problem of the reflection and refraction of an arbitrary electromagnetic wave at interfaces separating media of different dielectric properties. Several special problems of this type have been considered in detail by various authors, the first complete solution being the classical one of Sommerfeld [1] for the field of a vertical dipole above a plane earth. Hörschelmann [2], using the same method as Sommerfeld, obtained the solution for a horizontal dipole above the earth. Lurye [3] has considered a more general problem in calculating the field of a dipole above a dielectric slab resting on a plane earth of a large but finite conductivity. A two-dimensional version of this latter problem had been considered by Tai [4] who made the assumption that the conductivity of the earth was infinite.

All the above problems may be regarded as special cases of the more general problem of an arbitrary electromagnetic wave incident on a dielectric slab between two different dielectrics. This general problem will be considered in the following work. Clearly it will not be possible to enter into a detailed discussion of the properties of the solution in a manner comparable to that of the above authors, and no attempt is made to do this. It is, however, possible to obtain some fairly simple results concerning the form of the solution at large distances from the source.

The method used is an extension of that adopted by Copson [5] for considering the diffraction of a plane harmonic electromagnetic wave by an aperture in a perfectly reflecting plane. The components of the electric field which are in a direction tangential to the plane interfaces are constructed so that they are continuous across both planes. The components of the magnetic field are then expressed, using Green's functions, in terms of the tangential electric components on the interfaces. Integral equations are then obtained for these latter

components by applying the condition of continuity of the tangential components of the magnetic field across the planes. These integral equations are then solved by means of double Fourier transforms.

The method is also extended to cover the case of one surface deviating slightly from a plane. The only difference in this problem is that the boundary conditions of continuity are now to hold on a surface which is not quite plane. The method adopted is to consider the boundary conditions on the plane from which the actual surface deviates slightly. The effect of this approach is that the components tangential to this plane will now have known discontinuities across the plane; the formulation will otherwise be the same as for plane surfaces.

The general results for a nearly plane surface will, of necessity, be quite complicated, but simplification is afforded by considering special forms of the surface and of the incident field. In any particular problem it should be possible to adopt the method used by Rice [6] in considering a perfectly reflecting nearly plane surface and compute various averages of the field components, the average being taken over many different surfaces possessing the same statistical properties.

2. Mathematical formulation

It will be assumed that there are three dielectric media and that they occupy the regions 1, $z > 0$; 2, $-a < a < 0$; and 3, $z < -a$, respectively, where z denotes the normal to the planes (see Fig. 1). The x and y axes of a system of Cartesian coordinates are chosen parallel to the planes in such a manner that $O(x,y,z)$ is a right-handed system of axes. For simplicity it is assumed that the magnetic permeability is the same for all the media, and a time factor $e^{i\omega t}$ will be understood throughout the work.

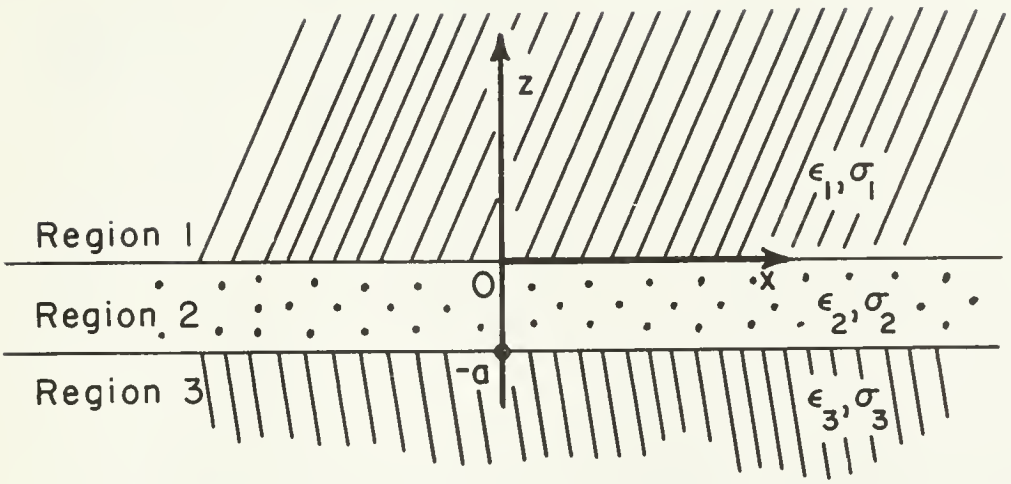

Figure 1

Maxwell's equations in each of the three regions (in normalized M.K.S. units) are

$$\text{(1)} \qquad \text{curl } \underline{H} = (\sigma_j + i\omega\epsilon_j\epsilon_o)\underline{E}, \qquad \text{curl } \underline{E} = -i\omega\mu_o\underline{H}, \qquad j = 1,2,3,$$

where $\underline{H}$ is the magnetic field, $\underline{E}$ the electric field, σ_j the conductivity, and ϵ_j the dielectric constant; ϵ_o, μ_o are the values of the dielectric constant and magnetic permeability in vacuo. It follows from equation (1) that all the Cartesian components of $\underline{E}$ and $\underline{H}$ satisfy Helmholtz's equation

$$\text{(2)} \qquad (\nabla^2 + k_j^2)\,\Psi = 0, \qquad j = 1,2,3,$$

where ∇^2 is the Laplacian operator and $k_j^2 = -i\omega\mu_o(\sigma_j + i\omega\epsilon_j\epsilon_o)$. Throughout the work it is assumed that k_j has a negative imaginary part (i.e., that the conductivity never vanishes); this assumption gives a simple form for the radiation condition at infinity, namely that all the components vanish exponentially with large distances. It also ensures that all the integrals occurring in the analysis are convergent.

On applying the method adopted by Copson it is fairly simple to obtain the following integral expressions for the Cartesian components of $\underline{E}$ and $\underline{H}$ in the region $z > 0$. The subscripts x,y,z in the formulas denote the component direction.

$$E_x = -\frac{\partial U_1}{\partial z}, \quad E_y = -\frac{\partial V_1}{\partial z}, \quad E_z = \frac{\partial U_1}{\partial x} + \frac{\partial V_1}{\partial y},$$

(3)

$$H_x = \frac{k_1^2 V_1}{i\omega\mu_o} - \frac{\partial W_1}{\partial x}, \qquad H_y = -\frac{k_1^2 U_1}{i\omega\mu_o} - \frac{\partial W_1}{\partial y}, \qquad H_z = -\frac{\partial W_1}{\partial z},$$

where

$$U_1 = \frac{1}{2\pi}\int \varphi_1 \mathcal{E}_x dx'dy', \quad V_1 = \frac{1}{2\pi}\int \varphi_1 \mathcal{E}_y dx'dy', \quad W_1 = \frac{1}{2\pi}\int \varphi_1 \mathcal{H}_z dx'dy'.$$

Here $\varphi = e^{-ik_1 r}/r$, where $r^2 = (x-x')^2 + (y-y')^2 + z^2$, and the script letters denote the values of the respective components on the plane $z' = 0$. Unless otherwise stated all integrals will be understood to be taken over the whole $(x'y')$-plane. By Maxwell's equations, the quantities U_1, V_1, W_1 are related by

$$W_1 = -\frac{1}{i\omega\mu_o}\left(\frac{\partial V_1}{\partial x} - \frac{\partial U_1}{\partial y}\right).$$

Equations (3) apply for the region $z < -a$ with only the difference that the signs of the expressions must be reversed and k_1 replaced by k_3. This gives for $z < -a$

$$E_x = \frac{\partial U_3}{\partial z}, \qquad E_y = \frac{\partial V_3}{\partial z}, \qquad E_z = -\frac{\partial U_3}{\partial x} - \frac{\partial V_3}{\partial y},$$

(4)

$$H_x = -\frac{k_3^2 V_3}{i\omega\mu_o} + \frac{\partial W_3}{\partial x}, \quad H_y = \frac{k_3^2 V_3}{i\omega\mu_o} + \frac{\partial W_3}{\partial y}, \quad H_z = \frac{\partial W_3}{\partial z},$$

where

$$U_3 = \frac{1}{2\pi}\int \varphi_3 \mathcal{E}'_x dx'dy', \quad V_3 = \frac{1}{2\pi}\int \varphi_3 \mathcal{E}'_y dx'dy', \quad W_3 = \frac{1}{2\pi}\int \varphi_3 \mathcal{H}'_z dx'dy'.$$

Here the primes denote values on $z' = -a - 0$, and $\phi_3 = \exp(-ik_3 r'/r')$, where $r'^2 = (x-x')^2 + (y-y')^2 + (z+a)^2$.

In order to calculate the field components in the region $-a < z < 0$ it is necessary to know the Green's functions $G_1(x-x', y-y', z,z')$, $G_2(x-x', y-y', z,z')$ for the region $-a < z < 0$, $|x|, |y| < \infty$. These are defined as solutions of

$$(\nabla^2 + k_2^2)G = -4\pi\delta(x-x')\delta(y-y')\delta(z-z')$$

such that $G_1 = 0$ for $z' = 0, -a$ and $\partial G_2/\partial z' = 0$ for $z' = 0, -a$; $\delta(x-x')$ is the Dirac delta-function. Both these functions have the common property of behaving like diverging waves at a large distance from $(x'y'z')$. The exact determination of the Green's functions is considered in the following section.

It is easy to show that the field components for region 2 are given by

$$(5) \qquad E_x = -\frac{1}{4\pi}\int\left\{\mathcal{E}_x''\left(\frac{\partial G_1}{\partial z'}\right)_{z'=0} - \mathcal{E}_x'''\left(\frac{\partial G_1}{\partial z'}\right)_{z'=-a}\right\} dx'dy',$$

$$(6) \qquad E_y = -\frac{1}{4\pi}\int\left\{\mathcal{E}_y''\left(\frac{\partial G_1}{\partial z'}\right)_{z'=0} - \mathcal{E}_y'''\left(\frac{\partial G_1}{\partial z'}\right)_{z'=-a}\right\} dx'dy',$$

$$(7) \qquad H_x = \frac{1}{4\pi}\int\left\{-\frac{k_2^2}{i\omega\mu_o}\left[(G_2)_{z'=0}\mathcal{E}_y'' - (G_2)_{z'=-a}\mathcal{E}_y'''\right]\right\} dx'dy'$$

$$+ \frac{1}{4\pi}\frac{\partial}{\partial x}\int\left[\mathcal{H}_z''(G_2)_{z'=0} - \mathcal{H}_z'''(G_2)_{z'=-a}\right] dx'dy'$$

$$(8) \qquad H_y = \frac{k_2^2}{i\omega\mu_o}\int\left[(G_2)_{z'=0}\mathcal{E}_x'' - (G_2)_{z'=-a}\mathcal{E}_x'''\right]dx'dy'$$

$$- \frac{1}{4\pi}\frac{\partial}{\partial y}\int\left[\mathcal{H}_z''(G_2)_{z'=0} - \mathcal{H}_z'''(G_2)_{z'=-a}\right]dx'dy',$$

where the triple primes denote values on $z = -a+0$ and the double primes denote values on $z = 0 - 0$. The expressions obtained for H_x and H_y by appli-

cation of Green's theorem to G_2 and $\mathcal{H}_x$ and $\mathcal{H}_y$ will actually involve $\partial \mathcal{H}_x/\partial z'$, $\partial \mathcal{H}_y/\partial z'$, but substitution from Maxwell's equations for these terms and some algebraic manipulation will yield equations (7) and (8). It should also be noted that the $\mathcal{H}_z$'s in these equations may be expressed in terms of x', y'-derivatives of the tangential electric components.

If there is an incident field $\underline{E}^i$, $\underline{H}^i$ in the region $z > 0$, then the total field may be written as $\underline{E}^i + \underline{E}^r$, $\underline{H}^i + \underline{H}^r$ in $z > 0$ and as $\underline{E}^t$, $\underline{H}^t$ in $z < 0$. It is necessary that E_x and E_y be continuous across $z = 0$, and hence

$$E_x^r = -E_x^i + E_x^t, \qquad E_y^r = -E_y^i + E_y^t \quad \text{on } z = 0.$$

If we write $\mathcal{E}_x$ and $\mathcal{E}_y$ for E_x^r, E_y^r on $z = 0$ then E_x^r, E_y^r at any point will be given by equation (3). The part of the integral in equation (3) involving $-E_x^i$, $-E_y^i$ on $z' = 0$ clearly corresponds to the reflected field in the region $z > 0$ when the plane is perfectly reflecting. Hence, if we use the superscript 0 to denote the field in $z > 0$ when the plane is perfectly reflecting, and write $\mathcal{E}_x''$, $\mathcal{E}_y''$ for E_x^t, E_y^t on the plane, the field components for $z > 0$ are given by

$$E_x = E_x^o - \frac{1}{2\pi}\frac{\partial}{\partial z}\int \varphi_1 \mathcal{E}_x'' \, dx'dy', \qquad E_y = E_y^o - \frac{1}{2\pi}\frac{\partial}{\partial z}\int \varphi_1 \mathcal{E}_y'' \, dx'dy',$$

(9)

$$E_z = E_z^o + \frac{1}{2\pi}\left\{\frac{\partial}{\partial x}\int \varphi_1 \mathcal{E}_x'' \, dx'dy' + \frac{\partial}{\partial y}\int \varphi_1 \mathcal{E}_y'' \, dx'dy'\right\},$$

$$H_z = H_z^o + \frac{1}{2\pi i\omega\mu_o}\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial x}\int \varphi_1 \mathcal{E}_y'' \, dx'dy' - \frac{\partial}{\partial y}\int \varphi_1 \mathcal{E}_x'' \, dx'dy'\right\},$$

$$H_x = H_x^o + \frac{1}{2\pi i\omega\mu_o}\left\{(k_1^2 + \frac{\partial^2}{\partial x^2})\int \varphi_1 \mathcal{E}_y'' \, dx'dy' - \frac{\partial^2}{\partial x\partial y}\int \varphi_1 \mathcal{E}_x'' \, dx'dy'\right\},$$

$$H_y = H_y^o - \frac{1}{2\pi i\omega\mu_o}\left\{(k_1^2 + \frac{\partial^2}{\partial y^2})\int \varphi_1 \mathcal{E}_x'' \, dx'dy' - \frac{\partial^2}{\partial x\partial y}\int \varphi_1 \mathcal{E}_y'' \, dx'dy'\right\}.$$

If we put $\mathcal{E}'_x = \mathcal{E}''_x$, $\mathcal{E}'_y = \mathcal{E}'''_y$, then equations (4) - (9) represent the solution of Maxwell's equations in all three regions with E_x, E_y continuous across $z = 0$, $z = -a$. It is clear that the condition of continuity of H_x, H_y, across the surfaces will give rise to four integral equations, but the actual equations will not be given here. In these equations the terms (H_x^o), (H_y^o) on $z = 0$ will occur and these two terms must be replaced by $2H_x^i$, $2H_y^i$ on $z = 0$. The solution of these equations is considered in Section 4.

3. The determination of the Green's functions

Both G_1, G_2 are solutions of

$$(\nabla^2 + k_2^2)G = -4\pi\delta(x-x')\delta(y-y')\delta(z-z').$$

The double Fourier transform g of G is defined by

$$g = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int G(t_1,t_2,z,z')\exp\left[-i(\alpha t_1 + \beta t_2)\right]dt_1dt_2 \,. \tag{10}$$

The behavior of G as $(x-x')^2 + (y-y')^2 \to \infty$ ensures that the integral of equation (10) converges for all real α and β. We also know that the differential equation for g is

$$\frac{d^2g}{dz^2} - \mu_2^2 g = -2\,\delta(z-z')$$

where $\mu_2^2 = \alpha^2 + \beta^2 - k_2^2$, and g is continuous at $z = z'$, but $\left(\frac{dg}{dz}\right)_{z=z'-0} - \left(\frac{dg}{dz}\right)_{z=z'+0} = 2$.

Thus we have the following expressions for g_1, g_2:

$$g_1 = -\frac{2}{\mu_2}\frac{\sinh \mu_2 z \sinh \mu_2(z'+a)}{\sinh \mu_2 a}, \qquad g_2 = \frac{2}{\mu_2}\frac{\cosh \mu_2(z'+a)\cosh \mu_2 z}{\sinh \mu_2 a}, \quad z \geq z';$$

$$g_1 = -\frac{2}{\mu_2}\frac{\sinh \mu_2 z' \sinh \mu_2(z+a)}{\sinh \mu_2 a}, \qquad g_2 = \frac{2}{\mu_2}\frac{\cosh \mu_2(z+a)\cosh \mu_2 z'}{\sinh \mu_2 a}, \quad z \leq z'.$$

We take that branch of the root μ_2 which reduces to ik_2 for $\alpha = \beta = 0$. It is also known that the double Fourier transform of $\phi_1 = \Psi_1$ may be written as

$$\Psi_1 = \frac{1}{\mu_1} \exp\left[-|z-z'|\mu_1\right] .$$

4. Solution of the integral equations

All the integral equations occurring in our problem are of the form

$$f(x,y) = \frac{1}{2\pi} \int\int g(x',y')h(x-x',y-y')dx'dy'. \tag{11}$$

Denoting the double Fourier transforms of the above functions by capital letters, we may write equation (11) as

$$F = GH,$$

as is well known. Now treating the integral equations occurring in the present analysis in this manner, we obtain the following equations:

$$2i\omega\mu_o H_x^{*i} = g\left[(\alpha^2-k_1^2)\,\Phi_1 + \Phi_2\,(\alpha^2-k_2^2)\right] - \alpha\beta f(\,\Phi_1 + \Phi_2) + g_1\,\Phi_4(k_2^2-\alpha^2) + \alpha\beta\,\Phi_4 f_1 , \tag{12}$$

$$2i\omega\mu_o H_y^{*i} = f\left[\Phi_1(k_1^2-\beta^2) + \Phi_2(k_2^2-\beta^2)\right] + \alpha\beta g(\,\Phi_1 + \Phi_2) + f_1\,\Phi_4(\beta^2-k_2^2) - \alpha\beta\,\Phi_4 g_1 , \tag{13}$$

$$g_1\left[(k_2^2-\alpha^2)\,\Phi_2 + (k_3^2-\alpha^2)\,\Phi_3\right] + \alpha\beta f_1\left[\Phi_2 + \Phi_3\right] = g\left[k_2^2-\alpha^2\right]\Phi_4 + \alpha\beta f\,\Phi_4 , \tag{14}$$

$$f_1\left[(\beta^2-k_2^2)\,\Phi_2 + (\beta^2-k_3^2)\,\Phi_3\right] - \alpha\beta g_1\left[\Phi_2 + \Phi_3\right] = f\,\Phi_4(\beta^2-k_2^2) - \alpha\beta g\,\Phi_4 , \tag{15}$$

where $\Phi_1 = 1/\mu_1$, $\Phi_3 = 1/\mu_3$, $\Phi_2 = (1/\mu_2)\coth \mu_2 a$, $\Phi_4 = (1/\mu_2)\operatorname{cosech} \mu_2 a$, and

H_x^{*i}, H_y^{*i}, f, g, f_1, g_1 are the Fourier transforms of $(H_x^i)_o$, $(H_y^i)_o$, $\mathcal{E}''_x$, $\mathcal{E}'_x$, $\mathcal{E}'_y$.

The solution of equations (12)-(15) has been obtained explicitly, but the rather complicated expressions obtained are not given here. In a later section it is shown that the far field depends on the values of the functions at one point in the $\alpha\beta$-plane, and it is possible that the best method of obtaining this far field is to solve equations (12)-(15) numerically for the particular values of α and β which are required.

5. The solution of special problems

In this section the results of the previous sections will be applied to the problems considered by Sommerfeld and Hörschelmann.

(a) Sommerfeld's problem

In this problem a and k_3 approach ∞. The incident field is due to a vertically polarized Hertzian oscillator, whose Fourier transforms are easily found to be

$$H_x^{*i} = \frac{k_1^2\beta}{\omega\mu_o\mu_1} e^{-\mu_1 h}, \qquad H_y^{*i} = -\frac{\alpha k_1^2}{\omega\mu_o\mu_1} e^{-\mu_1 h},$$

where h is the height of the dipole above the earth, $(z = 0)$.

If these expressions are substituted in equations (12) and (13) then the following expression is obtained for f:

$$f = \frac{2i\alpha k_1^2\mu_2}{(k_1^2\mu_2 + k_2^2\mu_1)} e^{-\mu_1 h} .$$

Application of the Fourier inversion theorem to the function U_1 given in equation (3) yields

$$U_1 = \frac{i}{2\pi} \int \frac{2\alpha k_1^2\mu_2 \exp\left[-\mu_1(z+h)\right] \exp\left[i(\alpha x+\beta y)\right] d\alpha d\beta}{\mu_1(k_1^2\mu_2 + k_2^2\mu_1)} . \tag{16}$$

Now we make the following substitutions in equation (16): $\alpha = \lambda\cos\theta$, $\beta = \lambda\sin\theta$, $y = \rho\sin\phi$, $x = \rho\cos\phi$. Then on application of the identity

$$\cos\phi\, J_1(\lambda\rho) = \frac{1}{2\pi i}\int_0^{2\pi} e^{i\lambda\rho\cos(\phi-\theta)}\cos\theta\, d\theta$$

it is seen that U_1 may be rewritten as

$$(17) \qquad U_1 = \frac{\partial}{\partial x}\left\{2k_1^2\int_0^{\infty} \frac{\sqrt{\lambda^2-k_2^2}\,\exp\left[-\sqrt{\lambda^2-k_1^2}(z+h)\right]J_0(\lambda\rho)\lambda\, d\lambda}{\sqrt{\lambda^2-k_1^2}\,(k_2^2\sqrt{\lambda^2-k_1^2}+k_1^2\sqrt{\lambda^2-k_2^2})}\right\}.$$

Equation (17) agrees with the result given by Sommerfeld.

(b) Hörschelmann's problem

This problem differs from that of Sommerfeld treated above only in that the polarization of the Hertzian oscilator is now horizontal. We have

$$H_x^{*i} = 0, \quad H_y^{*i} = k_1^2\exp\left[-\mu_1 h\right]i\omega\mu_0 .$$

If the expression for H_y^{*i} is substituted in equations (12), (13) and we proceed as in the previous problem, then an application of the integral representations for $J_0(\lambda\rho)$, $J_2(\lambda\rho)$ gives

$$(18) \qquad U_1 = -\int_0^{\infty} \frac{\lambda^3 k_1^2\exp\left[-\mu_1(z+h)\right]}{\mu_1(k_1^2\mu_2+k_2^2\mu_1)}\left[J_0(\lambda\rho)-J_2(\lambda\rho)\cos 2\phi\right]$$

$$+ 2k_1^2\int_0^{\infty} \frac{J_0(\lambda\rho)\exp\left[-\mu_1(z+h)\right]}{(\mu_1+\mu_2)\mu_1}\,\lambda\, d\lambda .$$

In equation (18) it is assumed that μ_1, μ_2 refer to $\sqrt{\lambda^2-k_1^2}$, $\sqrt{\lambda^2-k_2^2}$.

Hörschelmann in his solution used a Hertz vector $\underline{\Pi}$ of two components Π_x, Π_z, and the parts of $\underline{\Pi}$ reflected because of the imperfect conductivity of the earth were found to give

$$(19)\qquad \Pi_x = 2\int_0^\infty \frac{J_0(\lambda\rho)\exp\left[-\mu_1(z+h)\right]}{\mu_1+\mu_2}\,\lambda d\lambda,$$

$$\operatorname{div}\underline{\Pi} = -2\cos\phi \int_0^\infty \frac{J_1(\lambda\rho)\exp\left[-\mu_1(z+h)\right]k_1^2}{k_2^2\mu_1 + k_1^2\mu_2}\,\lambda d\lambda.$$

It is now easily verified that the field given by $E_x = k_1^2\Pi_x + \frac{\partial}{\partial x}(\operatorname{div}\underline{\Pi})$ and derived from equation (19) agrees with that given by $-\partial U_1/\partial z$, where U_1 is defined in equation (18).

In this particular problem the present approach has an advantage over that of Hörschelmann since in his solution, although only one component of the Hertz vector was present in the incident field, the total field requires two components of the Hertz vector. The boundary conditions at the interface would otherwise lead to a contradiction. The present formulation avoids all these difficulties.

6. The asymptotic forms of the field

In this section we shall consider the form of the field at large distances from the slab (for this work it will be assumed that the sources of the incident field are at a finite distance from the plane $z = 0$).

For convenience we shall replace x, y, z by x_1, x_2, x_3. Hence we have to consider the form, as $R^2 = x_1^2 + x_2^2 + x_3^2 \to 0$, of the following type of integral:

$$(20)\qquad \frac{1}{2\pi}\,\frac{\partial^s}{\partial x_i^r \partial x_j^\ell}\int \phi_1\, f(x',y')\,dx'dy'$$

where $r + \ell = s$.

For x', y' small,

$$\phi_1 \sim \frac{e^{-ik_1R}}{R} \exp\left[ik_1(x_1x' + x_2y')/R\right] + O(R^{-2})$$

and also

$$\frac{\partial^s}{\partial x_i^r \partial x_j^\ell}\left\{\frac{e^{-ik_1R}}{R}\exp\left[ik_1(x_1x'+x_2y')/R\right]\right\}$$

$$= \left(\frac{ik_1}{R}\right)^s x_i^r x_j^\ell \frac{e^{-ik_1R}}{R}\exp\left[ik_1(x_1x'+y_1y')/R\right] + O(R^{-2}).$$

If we assume that the main contribution to (20) for R large is from $(x'^2+y'^2)$ small, then the asymptotic form of (20) as $R \to \infty$ is

$$(21)\quad \frac{(ik_1)^s}{2\pi}\frac{x_i^r x_j^\ell}{R^s}\frac{e^{-ik_1R}}{R}\int\exp\left[ik_1(x'\cos\theta\cos\phi + y'\sin\theta\cos\phi)\right]dx'dy',$$

where $x_1 = R\cos\theta\cos\phi$, $x_2 = R\sin\theta\cos\phi$, $x_3 = R\sin\phi$. The integral in equation (21) is seen to represent 2π times the Fourier transform of f at $\alpha = -k_1\cos\theta\cos\phi$, $\beta = -k_1\sin\theta\cos\phi$.

If the above approach is applied to the following identity:

$$\frac{1}{2\pi}\int\frac{\partial\phi_1}{\partial z}H_x^i(x',y',0)dx'dy' \equiv H_x^i(x,y,-z), \qquad z > 0,$$

then we have

$$ik_1\sin\phi\,\frac{e^{-ik_1R}}{R}H_x^{*i}(-k_1\cos\theta\cos\phi,\ -k_1\sin\theta\cos\phi) = h_x,$$

where h_x is the radiation field of H_x at the image of the observation point in the plane $z = 0$. If the incident field is due to a source distribution contained in a finite volume then h_x may be regarded as the field at the observation point due to the image of the source distribution in the plane $z = 0$. Thus it is possible to express the total radiation field directly in terms of that of the incident field.

The present method of expanding the integrand gives the same result as that obtained by applying the saddle-point method for double integrals [7] to the integrals of the Fourier inversion formula.

Finally it should be noted that the present method is not applicable if z is small, i.e., if R is large and ϕ small, since in this case the exponential in the integrand of (20) becomes small and it is not possible to neglect the contributions to the integral from the region near $(x',y') = (x,y)$.

For the case of a plane earth the radiation field due to the imperfect conductivity is

$$\lambda E_x = -2i\omega\mu_o h_x k_1^2 \sin\theta\cos\theta\cos^2\phi \left[\frac{1}{k_1\sin\phi} + \frac{1}{\sqrt{k_2^2-k_1^2\cos^2\phi}}\right]$$

$$+2i\omega\mu_o h_y \left[-\frac{k_1(\cos^2\theta\cos^2\phi-1)}{\sin\phi} + \frac{(k_2^2-k_1^2\cos^2\theta\cos^2\phi)}{\sqrt{k_2^2-k_1^2\cos^2\phi}}\right],$$

$$\lambda E_y = -2i\omega\mu_o h_y k_1^2 \sin\theta\cos\theta\cos^2\phi \left[\frac{1}{k_1\sin\phi} + \frac{1}{\sqrt{k_2^2-k_1^2\cos^2\phi}}\right]$$

$$-2i\omega\mu_o h_x \left[\frac{k_1^2(1-\sin^2\theta\cos^2\phi)}{\sin\phi} + \frac{(k_2^2-k_1^2\sin^2\theta\cos^2\phi}{\sqrt{k_2^2-k_1^2\cos^2\phi}}\right],$$

$$\frac{-\lambda H_x\sin\phi}{2\omega\mu_o k_1} = \frac{(k_1^2-k_2^2)h_y\sin\theta\cos\theta\cos^2\phi}{\sqrt{k_2^2-k_1^2\cos^2\phi}} + h_x\left\{k_1\sin\phi - \frac{[k_1^2\sin^2\theta\cos^2\phi-k_2^2\cos^2\theta\cos^2\phi]}{\sqrt{k_2^2-k_1^2\cos^2\phi}}\right\},$$

$$\frac{-\lambda H_y\sin\phi}{2\omega\mu_o k_1} = h_y\left\{k_1\sin\phi - \frac{[k_1^2\sin^2\theta\cos^2\phi-k_2^2\cos^2\theta\cos^2\phi]}{\sqrt{k_2^2-k_1^2\cos^2\phi}}\right\} - h_x\frac{(k_1^2-k_2^2)\sin\theta\cos\theta\cos^2\phi}{\sqrt{k_2^2-k_1^2\cos^2\phi}},$$

where

$$\lambda = \frac{(k_1 \sin\phi + \sqrt{k_2^2 - k_1^2\cos^2\phi})(\sqrt{k_2^2 - k_1^2\cos^2\phi}\, k_1^2 + k_2^2 k_1 \sin\phi)}{\sin\phi\ \sqrt{k_2^2 - k_1^2\cos^2\phi}} .$$

7. Solution for one surface deviating slightly from a plane

In this section we consider the problem in which one interface, namely the one at $z = 0$, is not plane but defined by $z = \varepsilon h(x,y)$, where ε is small.

The solution for a plane interface will be denoted by the superscript (1), and the total solution is assumed to have the form

$$\underline{E}^{(1)} + \varepsilon\, \underline{E}^{(2)}, \qquad \underline{H}^{(1)} + \varepsilon\, \underline{H}^{(2)} .$$

The boundary conditions to be imposed on $z = \varepsilon h(x,y)$ are that the components of the electric and magnetic fields tangential to the surface be continuous across the surface. Hence to the first order in ε,

$$(22) \qquad E_{x1}^{(1)} + \varepsilon\, E_{x1}^{(2)} - \varepsilon \frac{\partial h}{\partial x} E_{z1}^{(1)} = E_{x2}^{(1)} + \varepsilon\, E_{x2}^{(2)} - \varepsilon \frac{\partial h}{\partial x} E_{z2}^{(1)} \quad \text{on } z = \varepsilon h(x,y);$$

a similar equation holds for E_y.

On expanding equation (22) in powers of ε and neglecting the terms of order greater than the first, we obtain

$$E_{x1}^{(2)} = E_{x2}^{(2)} + \frac{\partial h}{\partial z}(E_{x2}^{(1)} - E_{x1}^{(1)}) + \frac{\partial h}{\partial x}(E_{z1}^{(1)} - E_{z2}^{(1)}),$$

and similarly for $E_y^{(2)}$.

Let us denote by $\mathcal{E}_x^{(2)'}$, $\mathcal{E}_y^{(2)'}$ the values of $E_x^{(2)}$, $E_y^{(2)}$ on $z = 0 - 0$; then equations (5) - (8) hold for $E_x^{(2)}$ etc. in the region $-a < z < 0$ if $\mathcal{E}_x''$ and $\mathcal{E}_y''$ are replaced by $\mathcal{E}_x^{(2)''}$ and $\mathcal{E}_y^{(2)''}$. The field components $E_x(z)$ etc. in $z < -a$, will be of the same form as before except with the superscript (2) in each

expression. The field components in $z > 0$ will now be given by (3) with $\mathcal{E}_x$ and $\mathcal{E}_y$ replaced by

$$\mathcal{E}_x^{(2)'} + h \cdot \frac{\partial}{\partial z} (E_{x2}^{(1)} - E_{x1}^{(1)}) + (E_{z1}^{(1)} - E_{z2}^{(1)}) \frac{\partial h}{\partial z}$$

and the corresponding expression for $\mathcal{E}_y$. The boundary conditions on $H_x^{(2)}$, $H_y^{(2)}$ on $z = 0$ are

$$H_{x1}^{(2)} = H_{x2}^{(2)} + h \frac{\partial}{\partial z} (H_{x2}^{(1)} - H_{x1}^{(1)}) \tag{23}$$

$$H_{y1}^{(2)} = H_{y2}^{(2)} + h \frac{\partial}{\partial z} (H_{y2}^{(1)} - H_{y1}^{(1)}) \, . \tag{24}$$

Equations (23), (24) contain no terms in $\partial h/\partial x$, $\partial h/\partial y$, since by our assumption of unit permeability $H_z^{(1)}$ is continuous at $z = 0$.

If the above changes are carried out in the formulas and the boundary conditions (23), (24) are applied, then a set of integral equations will be obtained for $\mathcal{E}_x^{(2)}$ similar to those obtained in the plane case. The only difference is that the incident field in the latter equations will have to be replaced by functions involving the field components for a plane surface. The explicit form for these functions as Faltung integrals has been obtained, but in view of their complexity they will not be given here. Some simplification would be afforded if the surface were assumed to have a special form, and it would be possible, with a considerable amount of labor, to carry out an analysis for any particular given field using an approach similar to that of Rice.

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